SIMPLE COMPONENTS OF SEMISIMPLE GROUP ALGEBRAS OF METACYCLIC GROUPS

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Abstract. *Let G be a metacyclic group and K be such a field that the group algebra KG is semisimple. Several authors have examined the algebra KG and have obtained important results in two particular cases for the field K: when $K = Q$ is the field of rational numbers and when K is finite field with characteristic, coprime to the order of G. The contribution of our work is in the determination up to isomorphism of the simple components of the algebra KG where the aforementioned limitations over the field K have been overcome. In the last section we give illustrative examples of a specific group G and field K.*

Key words: metacyclic group; finite semisimple group algebra; minimal orthogonal idempotent; simple component; Wedderburn decomposition.

1. Introduction

Let K be a field and G be such a finite group that the group algebra KG is semisimple. In order to describe the structure of the algebra KG it is sufficient to find one complete system of minimal central orthogonal idempotents of KG and the Wedderburn decomposition of KG [8, Theorem 1.4.4]. (The above is true because every such idempotent e generate a minimal two-sided ideal KGe of the algebra KG, which is called a simple component of KG and is isomorphic to direct summand in the Wedderburn decomposition of the group algebra.) This problem is of interest due to its applications in both pure and applied algebra.

In the last few years much researches has been done to describe the structure of the semisimple group algebra KG of the finite group G in two important special cases: when $K = Q$ is the field of rational numbers and when K is finite field with characteristic, coprime to the order of G.

In the special case when $K = Q$, Olivieri, del Rio and Simon gives a description of the primitive central idempotents and the corresponding simple components of the semisimple group algebra QG, as well as the Wedderburn decomposition of the algebra QG, when G is abelian-by-supersolvable group [9]. In [10] they examine when two simple components of the algebra QG of a metacyclic group G are isomorphic. These results have been used by Bakshi and Maheshwary, when G is a normally monomial group [4], as well as by Bakshi and Kaur [3], when G is a group such that all the subgroups and quotient groups of G satisfy the following property: either it is abelian or it contains a non-central abelian normal subgroup.

In the special case, when K is finite field with characteristic, coprime to the order of the group G, Broche and del Rio give a description of the primitive central idempotents and structure of the corresponding simple components of the semisimple group algebra KG, when G is an abelian-by-supersolvable group [5]. Bakshi, Gupta and Passi specify the structure of the algebra KG, when G is a metacyclic group [1] and when G is a metabelian group [2].

The results shown for the semisimple group algebra KG of a metacyclic group G were obtained by the different authors with significant restrictions for the field K. Namely, when $K = Q$ is the field of rational numbers or when K is a finite field with characteristic, coprime to the order of G. The additional value of this work compared to those described above is that we overcome the aforementioned restrictions for the field K. We find the simple components of the algebra KG in the most general case, i.e. when K is an arbitrary field (it sufficient for KG to be a semisimple algebra).

This article is a continuation of our previous work [6], in which we solve the problem of finding a complete system of minimal central orthogonal idempotents of the semisimple group algebra KG in the most general case possible, i.e. when K is an arbitrary field [6, Theorem 8]. In Section 2 we get some preliminary results for the algebra KG on the condition that K is a field of decomposition of the metacyclic group G. We find the dimension of the ideal KGe as K-algebra (Theorem 2.2) and describe the Wedderburn decomposition of the algebra KG (Theorem 2.3). In Section 3, without imposing limitations over the field K, we determine up to isomophism the simple components of the algebra KG (Theorem 3.1). Our study concludes with Section 4 where we illustrate the achieved results with specific examples.

2. Finding the simple components and determining of the Wedderburn decomposition of the semisimple group algebra KG, when K is a field of decomposition of the metacyclic group G

Let G be a finite metacyclic group, determined by the conditions [7, Theorem 9.4.3]:

$$\begin{array}{l} G = \left\langle a, b \,\middle|\, a^n = 1,\, b^m = 1,\, a^{-1}ba = b^r \right\rangle, \\ \text{where } \left(r\left(r-1\right), m\right) = 1 \text{ and } r^n \equiv 1 \left(\operatorname{mod} m\right). \end{array} \tag{1}$$

Bearing in mind the conditions (1), in this article we will permanently use the notation introduced in [6].

Let K be a field of decomposition of the metacyclic group G with determining conditions (1). Then according to [6, Theorem 6] the minimal central orthogonal idempotents of the semisimple group algebra KG for a fixed divisor d of m are $e_{dij} = \alpha_{di} B_{dj}$, where

$$\begin{array}{l} \alpha_{di} = \dfrac{k}{n} \displaystyle\sum_{s=0}^{l-1} \left(\frac{a^k}{\varepsilon^i}\right)^s \quad \text{for } i = 0,\, 1,\, \ldots,\, l-1, \\ B_{dj} = \displaystyle\sum_{\mu=0}^{k-1} \beta_{djr^\mu} = \sum_{\mu=0}^{k-1} \left(\frac{1}{m} \sum_{s_1=0}^{m-1} \left(\frac{b}{\eta^{jr^\mu}}\right)^{s_1}\right) \quad \text{for } j \in J. \end{array} \tag{2}$$

Let $e = \frac{k}{mn} \left(\sum\limits_{s=0}^{l-1} \frac{a^{ks}}{\varepsilon^s}\right) \left(\sum\limits_{\mu=0}^{k-1} \sum\limits_{s_1=0}^{m-1} \frac{b^{s_1}}{\eta^{s_1 r^\mu}}\right)$ be an arbitrary idempotent of the semisimple group algebra KG. If $\alpha = \frac{k}{n} \left(\sum\limits_{s=0}^{l-1} \frac{a^{ks}}{\varepsilon^s}\right)$ and $\beta_{r^\mu} = \frac{1}{m} \left(\sum\limits_{s_1=0}^{m-1} \frac{b^{s_1}}{\eta^{s_1 r^\mu}}\right)$, then $e = \sum\limits_{\mu=0}^{k-1} e_\mu$, where $e_\mu = \alpha \beta_{r^\mu}$.

Theorem 2.1. *The element be from the ideal KGe of the semisimple group algebra KG is a root of the polynomial $f(x) = \prod\limits_{\mu=0}^{k-1} \left(x - \eta^{r^\mu} e\right)$ of degree k.*

Proof. First we will prove that the equality $be_\mu = \eta^{r^\mu} e_\mu$ holds. Indeed, as α is a central element of the semisimple group algebra KG, then $be_\mu = \frac{\alpha}{m} \left(\sum\limits_{s_1=0}^{m-1} \frac{b^{s_1+1}}{\eta^{s_1 r^\mu}}\right) = \eta^{r^\mu} \frac{\alpha}{m} \sum\limits_{s_1=0}^{m-1} \frac{b^{s_1+1}}{\eta^{(s_1+1) r^\mu}} = \eta^{r^\mu} \alpha \beta_{r^\mu} = \eta^{r^\mu} e_\mu$. Let us calculate

$f(be)$:

$$f(be)=\prod_{\mu=0}^{k-1}\left(b-\eta^{r^{\mu}}\right)e=\prod_{\mu=0}^{k-1}\left(b-\eta^{r^{\mu}}\right)\sum_{\nu=0}^{k-1}e_{\nu}=\sum_{\nu=0}^{k-1}\prod_{\mu=0}^{k-1}\left(b-\eta^{r^{\mu}}\right)e_{\nu}.$$

Given $\mu=\nu$, from the initially proved equality we get $\left(b-\eta^{r^{\mu}}\right)e_{\mu}=0$, from where it follows $f(be)=0$.

□

Theorem 2.2. *The dimension of the ideal KGe as K-algebra is k^2.*

Proof. From Theorem 2.1 follows that the highest exponent of the element b which is contained in the ideal KGe of the algebra KG, is $k-1$. It is easy to see that the highest exponent of the element a, which is contained in the ideal KGe, is also $k-1$. Therefore $\dim_K KGe\leq k^2$. So the semisimple group algebra KG decomposes into a direct sum of ideals of the kind KGe and each of them has a dimension of at most k^2 over the field K. According to [6, comment after Theorem 7] for a fixed d the number of these ideals is $\frac{n\phi(d)}{k^2}$, where $\phi(x)$ is Euler's function. Therefore, $nm=\dim_K KG=\sum\limits_{e}\dim_K KGe\leq\sum\limits_{d/m}k^2\frac{n\phi(d)}{k^2}=n\sum\limits_{d/m}\phi(d)=nm$, i.e. $\dim_K KGe=k^2$.

□

Proven in Theorem 2.2 means that the ideal KGe of the semisimple group algebra KG is isomorphic to the matrix algebra $M_k(K)$ of the $k\times k$ matrices over the field K.

Bearing in mind that for a fixed d the number of the ideals of the kind KGe is $\frac{n\phi(d)}{k^2}$, we obtain the following:

Theorem 2.3. *(Wedderburn decomposition). Let G be a metacyclic group with determining conditions (1) and K be a field of decomposition of G, whose characteristic does not divide the order of the group. Then the Wedderburn decomposition of the semisimple group algebra KG is*

$$KG\cong\sum_{d/m}\oplus\frac{n\phi(d)}{k^2}M_k(K).$$

3. Finding the simple components of the semisimple group algebra KG of the metacyclic group G, when K is an arbitrary field

Let K be a field, whose characteristic does not divide the order of the finite group G. According to the Maschke's Theorem the group algebra KG is semisimple [11, §3.6]. Then from the Wedderburn Theorem it follows that the algebra KG decomposes in a direct sum of minimal two-sided ideals generated by minimal central orthogonal idempotents of KG [11, §3.5]. From [7, Theorem 16.5.2 and Lemma 16.5.2] holds:

Theorem 3.1. *Let K be a field, whose characteristic does not divide the order of the finite group G. The ideal KGe of the semisimple group algebra KG is isomorphic to the matrix algebra $M_k(D)$, where $D = e_1^* KGe_1^*$ for some minimal idempotent e_1^* of the ideal KGe of the algebra KG.*

4. Examples of finding the simple components and determining of the Wedderburn decomposition of the semisimple group algebra KG of the metacyclic group G

Example 4.1. *Let G be metacyclic group, generated by the elements a and b with determining conditions $a^8 = 1$, $b^{15} = 1$, $a^{-1}ba = b^2$ and K is a field of decomposition of the group G, i.e. K contains primitive 8-th root of 1 and primitive 15-th root of 1. We will find the simple components and determine the Wedderburn decomposition of the semisimple group algebra KG.*

First case. *For $d = 1$, $k = 1$ the simple components of the algebra KG are isomorphic to the field K and we get $K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K$.*

Second case. *For $d = 3$, $k = 2$ the simple components of KG are isomorphic to the matrix ring $M_2(K)$ and we get $M_2(K) \oplus M_2(K) \oplus M_2(K) \oplus M_2(K)$.*

Third case. *For $d = 5$, $k = 4$ the simple components of the algebra KG are isomorphic to the matrix ring $M_4(K)$ and we get $M_4(K) \oplus M_4(K)$.*

Fourth case. *For $d = 15$, $k = 4$ the simple components of the algebra KG are isomorphic to the matrix ring $M_4(K)$ and we get $M_4(K) \oplus M_4(K) \oplus M_4(K) \oplus M_4(K)$.*

Finally, the Wedderburn decomposition of the semisimple group al-

gebra KG of the metacyclic group G is:

$$\begin{aligned} KG \cong & K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus \\ & M_2(K) \oplus M_2(K) \oplus M_2(K) \oplus M_2(K) \oplus \\ & M_4(K) \oplus M_4(K) \oplus M_4(K) \oplus M_4(K) \oplus M_4(K) \oplus M_4(K)\,. \end{aligned}$$

Example 4.2. *We consider the semisimple group algebra QG of the metacyclic group G of Example 4.1. Here the field Q of the rational numbers is not a field of decomposition of the group G, because it does not contain a primitive 8th root of 1 and a primitive 15th root of 1.*

First case. *For $d = 1$, $k = 1$ the minimal central orthogonal idempotents of the algebra QG are $e_{1j1} = \alpha'_{1j} B_{11}$ for $j = 0, 1, 2, 3$, where*

$$\begin{aligned} \alpha'_{10} &= \frac{1}{8}(1+a)(1+a^2)(1+a^4), \\ \alpha'_{11} &= \frac{1}{2}(1-a)(1+a)(1+a^2), \\ \alpha'_{12} &= \frac{1}{4}(1-a)(1+a)(1+a^4), \\ \alpha'_{13} &= \frac{1}{8}(1-a)(1+a^2)(1+a^4), \\ B_{11} &= \frac{1}{15}(1+b+b^2+...+b^{14}). \end{aligned}$$

If ε is a primitive 8th root of 1 then for the simple components of the semisimple group algebra QG we get $QGe_{101} \cong QGe_{131} \cong Q$; $QGe_{111} \cong Q(\varepsilon)$; $QGe_{121} \cong Q(i)$. Then $\sum_{j=0}^{3} QGe_{1j1} \cong Q \oplus Q \oplus Q(\varepsilon) \oplus Q(i)$.

Second case. *For $d = 3$, $k = 2$ the minimal central orthogonal idempotents of the algebra QG are $e_{3j1} = \alpha'_{3j} B_{31}$ for $j = 0, 1, 2$, where*

$$\begin{aligned} \alpha'_{30} &= \frac{1}{4}(1+a^2)(1+a^4), \\ \alpha'_{31} &= \frac{1}{2}(1-a)(1+a)(1+a^2), \\ \alpha'_{32} &= \frac{1}{4}(1-a)(1+a)(1+a^4), \\ B_{31} &= \frac{1}{15}(2-b-b^2)(1+b^3+b^6+b^9+b^{12}). \end{aligned}$$

In this case one of the simple components of the algebra QG is

$QGe_{301} \cong M_2(Q)$. *For the other two simple components of the algebra* QG *we will prove that* $QGe_{311} \cong M_2(Q(i))$ *and* $QGe_{321} \cong \left(\frac{-1,-3}{Q}\right)$, *where* $\left(\frac{-1,-3}{Q}\right)$ *is a division ring of the quaternions over* Q.

First we will prove that $QGe_{311} \cong M_2(Q(i))$. *We will denote* $e = e_{311}$. *The ideal* QGe *is determined by the conditions*

$$a^4e = -e, \quad b^2e + be + e = 0, \quad a^{-1}bae = b^2e.$$

We consider the map $\phi : QGe \to M_2(Q(i))$, *for which* $\phi(ae) = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$, $\phi(be) = \begin{pmatrix} i & 1 \\ -i & -1-i \end{pmatrix}$. *It is easy to see that* ϕ *is an isomorphism and that* $QGe \cong M_2(Q(i))$.

We will prove that $QGe_{321} \cong \left(\frac{-1,-3}{Q}\right)$, *where* $\left(\frac{-1,-3}{Q}\right)$ *is the division ring of quaternions over* Q. *We denote*

$$e = e_{321} = \frac{1}{60}\left(1-a^2\right)\left(1+a^4\right)(2-b-b^2)(1+b^3+b^6+b^9+b^{12}).$$

It is to see that the ideal QGe *is determined by the conditions* $a^2e = -e$, $b^2e + be + e = 0$, $a^{-1}bae = b^2e$. *Let* $ce = 2be + e$ *be a new generator of* QGe. *Then* $c^2e = -3e$ *and* $a^{-1}cae = 2a^{-1}bae + e = -2be - 2e + e = -ce$, *from where* $cae = -ace$. *So* QGe *is isomorphic to the ring with determining conditions* $(ae)^2 = -e$, $c^2e = -3e$, $cae = -ace$. *But those are also determining conditions of the division ring* $\left(\frac{-1,-3}{Q}\right)$ *of the quaternions over* Q *(see [11, §1.6., Definition]). Therefore* $QGe_{321} \cong \left(\frac{-1,-3}{Q}\right)$.

Finally for the second case, $\sum_{j=0}^{2} QGe_{3j1} \cong M_2(Q) \oplus M_2(Q(i)) \oplus \left(\frac{-1,-3}{Q}\right)$.

Third case. *For* $d = 5$, $k = 4$ *the minimal central orthogonal idempotents of the algebra* QG *are* $e_{5j1} = \alpha_{5j}B_{51}$ *for* $j = 1, 2$, *where*

$$\alpha_{51} = \frac{1}{2}(1+a^4),$$
$$\alpha_{52} = \frac{1}{2}(1-a^4),$$
$$B_{51} = \frac{1}{15}(4-b-b^2-b^3-b^4)(1+b^5+b^{10}).$$

Then the simple components of the algebra QG are $QGe_{511} \cong M_4(Q)$ and $QGe_{521} \cong D_{16}$, where D is a division ring with dimension 16 over Q with determining conditions $a^4e = -e$, $b^4e + b^3e + b^2e + be + e = 0$, $a^{-1}bae = b^2e$.

We will prove that $QGe_{511} \cong M_4(Q)$, where

$$e_{511} = \frac{1}{2}(1+a^4)(4-b-b^2-b^3-b^4)(1+b^5+b^{10}).$$

The equation $(1+b+b^2+b^3+b^4)e_{511} = 0$ holds. The element $e' = \frac{1}{4}(1+a)(1+a^2)e_{511}$ is minimal non-central idempotent of QGe_{511}. It is easy to verify that $e'a^ie' = e' \in Qe' \cong Q$, $e'b^je' = -\frac{1}{4}e' \in Qe'$ and $e'a^ib^je' = -\frac{1}{4}e' \in Qe'$ for $i = 0, 1, ..., 7$ and $j = 0, 1, ..., 14$. Therefore $e'QGe' \cong Q$ and then $QGe_{511} \cong M_4(Q)$.

The proof of the isomorphism $QGe_{521} \cong D_{16}$ is standard.

Finally in the third case we obtain $QGe_{511} \oplus QGe_{521} \cong M_4(Q) \oplus D_{16}$.

Fourth case. *For $d = 15$, $k = 4$ the minimal central orthogonal idempotents of the algebra QG are $e_{15j1} = \alpha_{15j}B'_{151}$ for $j = 1, 2$, where*

$$\begin{aligned}
\alpha_{151} &= \frac{1}{2}(1+a^4),\\
\alpha_{152} &= \frac{1}{2}(1-a^4),\\
B'_{151} &= \frac{1}{15}(8+b+b^2-2b^3+b^4-4b^5-2b^6\\
&\quad +b^7+b^8-2b^9-4b^{10}+b^{11}-2b^{12}+b^{13}+b^{14}).
\end{aligned}$$

Let $\lambda = \eta+\eta^2+\eta^4+\eta^8$ and η be a primitive 15th root of the 1. In this case the simple components of the algebra QG are $QGe_{1511} \cong M_4(Q(\lambda))$ and $QGe_{1521} \cong D'_{16}$. Here D'_{16} is a division ring with dimension 16 over $Q(\lambda)$ and has the following determining conditions: $a^4e_{1521} = -e_{1521}$, $(1 - b + b^3 - b^4 + b^5 - b^7 + b^8)e_{1521} = 0$, $a^{-1}bae_{1521} = b^2e_{1521}$.

We will prove that $QGe_{1511} \cong M_4(Q(\lambda))$. The equations hold:

$$a^4e_{1511} = e_{1511} \quad \textit{and} \quad (c^2-c+4)e_{1511} = 0,$$

where $c = b+b^2+b^4+b^8$ is a central element of QG. Obviously $Q(c) \cong Q(\lambda)$ and $\dim_Q Q(c) = 2$. The element $e'' = \frac{1}{4}(1+a)(1+a^2)e_{1511}$ is non-central *minimal idempotent of QGe_{1511}. It is easy to see that*

$$e''a^ie'' = e'' \in Q(c)e'' \cong Q(c),$$
$$e''b^je'' = -\frac{c}{4}e'' \in Q(c)e'',$$
$$e''a^ib^je'' = -\frac{c}{4}e'' \in Q(c)e'' \quad \text{for } i = 0, 1, ..., 7, \text{ and } j = 0, 1, ..., 14.$$

Therefore $e''QGe'' \cong Q(\lambda)$ and $QGe_{1521} \cong M_4\left(Q(\lambda)\right)$.

In this case we obtain $QGe_{1511} \oplus QGe_{1521} \cong M_4\left(Q(\lambda)\right) \oplus D'_{16}$.

Finally for the second example the Wedderburn decomposition of the semisimple group algebra QG of the metacyclic group G is:

$$QG \cong Q \oplus Q \oplus Q\left(\varepsilon\right) \oplus Q\left(i\right) \oplus M_2(Q) \oplus M_2(Q\left(i\right)) \oplus \left(\frac{-1,-3}{Q}\right) \oplus M_4\left(Q\right) \oplus D_{16} \oplus M_4\left(Q\left(\lambda\right)\right) \oplus D'_{16}.$$

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Bibliography

[1] G. Bakshi, S. Gupta, I. Passi, The structure of finite semisimple metacyclic group algebras, *J Ramanujan Math Soc*, Vol. **28**, (2013), 141–158, ISSN: 0970-1249.

[2] G. Bakshi, S. Gupta, I. Passi, The algebraic structure of finite metabelian group algebras, *Comm Alg*, (2015), Vol. **43**, 2240–2257, ISSN: 0092-7872, DOI: `https://doi.org/10.1080/00927872.2014.888566`.

[3] G. Bakshi, G. Kaur, Character Triples and Shoda Pairs, *J Algebr*a, (2017), Vol. 491, 447–473, ISSN: 0021-8693, DOI: `https://doi.org/10.1016/j.jalgebra.2017.08.016`.

[4] G. Bakshi, S. Maheshwary, The rational group algebra of a normally monomial group, *J Pure Appl Alg*, (2014), Vol. 218, 1583–1593, ISSN: 0022-4049, DOI: `https://doi.org/10.1016/j.jpaa.2013.12.010`.

[5] O. Broche, A. del Rio, Wedderburn decomposition of finite group algebras, *Finite Fields Appl*, (2007), Vol. 13, 71–79, ISSN: 1071-5797, DOI: `https://doi.org/10.1016/j.ffa.2005.08.002`.

[6] I. Gradeva, Y. Epitropov, Minimal central orthogonal idempotents of semisimple group algebras of metacyclic groups, *Houston J Mathematics*, (2019), Vol. 45, 965–977, ISSN 0362-1588.

[7] M. Hall, *The Theory of Groups*, American Mathematical Soc., (1999), ISBN: 978-0821819678.

[8] I. Herstein, *Noncommutative rings*, Cambridge University Press, (2005), ISBN: 978-0883850398.

[9] A. Olivieri, A. del Rio, J. Simon, On monomial characters and central idempotents of rational group algebras, *Comm Algebra*, (2004), Vol. 32, 1531–1550, ISSN: 0092-7872, DOI: `https://doi.org/10.1081/AGB-120028797`.

[10] A. Olivieri, A. del Rio, J. Simon, The group of automorphisms of the rational group algebra of a finite metacyclic group, *Comm Algebra*, (2006), Vol. 34, 3543–3567, ISSN: 0092-7872, DOI: `https://doi.org/10.1080/00927870600796136`.

[11] R. Pierce, *Associative Algebras*, Springer, (2012), ISBN: 978-1475701630.

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