MEMORY, THINKING AND TEACHING IN MATHEMATICS

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Abstract. *This article offers some tools for the development of students' memory and thinking using different ways of teaching in mathematics. We have considered tasks in solving which require qualities such as resourcefulness, flexibility, transfer of knowledge in new conditions, etc.*

Key words: memory, thinking, important algebraic formulas.

It is known that there are millions of scientists in the world who contribute to the rapid development of science. Its turbulent development consists in several regularities – expanding, deepening, integrating and morally ageing knowledge [9]. The expansion of knowledge leads to the increasing of their volume, but it is not related with increasing of the human ability to remember and permanently store it.

We will note that creative thinking is one of the important moments in the preparation of young people. Their success depends from the relationship between memory and thinking at the leadership role of awareness and logical thinking. Therefore, still at school, students have to arm themselves with reliable methods to discover alone and process the information they are interested in. In order to be processed, the information must be memorized by the students. The degree of memorization depends from understanding [1, 5, 6]. The question arises: what are the means by which we will establish that a theory has been mastered with understanding? A popular tool are tasks solved with the help of a given knowledge. Indeed, on the way in which students solve a task can be judged for the depth of the acquired theoretical material, their ability to navigate in it, their ability to apply the knowledge [3, 5, 6].

We will note that solving tasks is a targeted and conscious activity. The indications inherent in forming skills for solving tasks that the scientific literature states are: occurrence of a search situation and selection of the necessary way of solving; existence of a logical connection between the separate parts of the solution; striving not to lose the actual content of the task; carrying out control for the correct implementation of the actions.

In the school course of study, students use a variety of important algebraic formulas. The first formulas that they study are:

$$(a+b)^2 = a^2 + 2ab + b^2, \tag{1}$$

$$(a-b)^2 = a^2 - 2ab + b^2, \tag{2}$$

$$a^2 - b^2 = (a+b)(a-b). \tag{3}$$

Then they study the formulas:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \tag{4}$$

which can be written in the form

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b) \tag{5}$$

and

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right). \tag{6}$$

At the next stage of training applying formula (3) they use the following entries

$$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}, \tag{7}$$

$$\sqrt{a} + \sqrt{b} = \frac{a-b}{\sqrt{a}-\sqrt{b}}, \tag{8}$$

where $a \geq 0$, $b \geq 0$, $\sqrt{a} - \sqrt{b} \neq 0$.

Formula

$$a^4 + b^4 = \left(a^2 + b^2\right)^2 - 2a^2b^2 = \left[(a+b)^2 - 2ab\right]^2 - 2a^2b^2 \tag{9}$$

is not studied at school, but is presented to students who are interested in mathematics in the additional elective hours. By looking at each of these formulas, we can use them for a different purpose.

The use of the various records that are applied in solving tasks contributes to their creative absorption. For example, a formula (1) can be used to raising square of a binomial, i.e. from left to right. It can also be applied from right to left for factoring of the expressions from this type. All this is included in the textbooks in the school mathematics course. However, this formula can be written in the form

$$a^2 + b^2 = (a+b)^2 - 2ab. \tag{10}$$

Thus written, it can be used to find the sum of the squares of two expressions. In this work, we will illustrate the use of different records of the above formulas in solving some types of problems. We will apply the formula (10) to solve the following task.

Task 1. Solve the equation

$$\left(\frac{x}{x-2}\right)^2+\left(\frac{x}{x+2}\right)^2=2.$$

Solution: The domain of the given equation is $x\neq\pm2$. Using the formula (10) the equation can be written in the form

$$\left(\frac{x}{x-2}+\frac{x}{x+2}\right)^2-2.\frac{x}{x-2}.\frac{x}{x+2}=2$$

or

$$\left(\frac{2x^2}{x^2-4}\right)^2-2.\frac{x^2}{x^2-4}=2.$$

Then after substituting $\frac{2x^2}{x^2-4}=y$ we get the quadratic equation $y^2-y-2=0$, whose roots are $y_1=-1$ and $y_2=2$. Since $\frac{2x^2}{x^2-4}=y$ we look at two equations:

1. If $y=-1$, then $\frac{2x^2}{x^2-4}=-1\Leftrightarrow 3^2=4$ or $x_{1/2}=\pm\frac{2\sqrt{3}}{3}$, which belong to the domain of our equation.
2. If $y=2$, that $\frac{2x^2}{x^2-4}=2$. But this equation has no roots. Therefore, the solutions of the given equation are $x_{1/2}=\pm\frac{2\sqrt{3}}{3}$.

Elements in the considered problem are the expressions $\frac{x}{x-2}$ and $\frac{x}{x+2}$, the relations between them are the sum, the product, the sum of squares. Here, the student must select one of the records of the formula, which will lead to a rational solution of the problem.

The formula (2) can be written as

$$a^2+b^2=(a-b)^2+2ab. \tag{11}$$

We will apply it to solve the following tasks.

Task 2. Solve the equation

$$x^2 + \frac{81x^2}{(x+9)^2} = 40.$$

Solution: The domain of the given equation is $x \neq -9$. Applying the formula (11) we obtain

$$\left(x - \frac{9x}{x+9}\right)^2 + \frac{18x^2}{x+9} = \left(\frac{x^2}{x+9}\right)^2 + \frac{18x^2}{x+9} = 40.$$

After substituting $\frac{x^2}{x+9} = y$ we get the quadratic equation $y^2+18y-40=0$, whose roots are $y_1 = 2$ and $y_2 = -20$. After taking the values for y from the substitution to be the roots of the quadratic equation, we find that the given equation has roots $1 \pm \sqrt{19}$.

Task 3. Solve the equation

$$10x^2(x-2)^2 = 9\left(x^2 + (x-2)^2\right).$$

Solution: In the solution of this equation, we will use again the formula (11). Then we have

$$10x^2(x-2)^2 = 9\left((x-(x-2))^2 + 2x\,(x-2)\right)$$

or

$$5x^2(x-2)^2 = 9\left(2 + x\,(x-2)\right).$$

After substituting $x\,(x-2) = y$ we get the quadratic equation $5y^2 - 9y - 18 = 0$, whose roots are $y_1 = 3$ and $y_2 = -\frac{6}{5}$. From the substitution we find that the given equation has roots the numbers 3 and -1.

Formula (3) can be written as

$$a + b = \frac{a^2 - b^2}{a - b}, \tag{12}$$

where $a \neq b$ and

$$a - b = \frac{a^2 - b^2}{a + b}, \tag{13}$$

where $a \neq -b$. We will illustrate their applications in solving the next tasks.

Task 4. Solve the system of equations

$$\left| \begin{array}{l} \sqrt{x+\sqrt{y}}+\sqrt{x-\sqrt{y}}=2 \\ \sqrt{y+\sqrt{x}}-\sqrt{y-\sqrt{x}}=1 \end{array} \right. .$$

Solution: I way. We consider the first equation of the given system and apply the formula (12) for it. Then

$$\begin{aligned} \sqrt{x+\sqrt{y}}+\sqrt{x-\sqrt{y}} &= \frac{\left(\sqrt{x+\sqrt{y}}\right)^2-\left(\sqrt{x-\sqrt{y}}\right)^2}{\sqrt{x+\sqrt{y}}-\sqrt{x-\sqrt{y}}} \\ &= \frac{2\sqrt{y}}{\sqrt{x+\sqrt{y}}-\sqrt{x-\sqrt{y}}}=2 \end{aligned}$$

or $\sqrt{x+\sqrt{y}}-\sqrt{x-\sqrt{y}}=\sqrt{y}$. We get the system

$$\left| \begin{array}{l} \sqrt{x+\sqrt{y}}+\sqrt{x-\sqrt{y}}=2 \\ \sqrt{x+\sqrt{y}}-\sqrt{x-\sqrt{y}}=\sqrt{y} \end{array} \right. ,$$

and then we add both equations and find that $2\sqrt{x+\sqrt{y}}=\sqrt{y}+2$. After raising to a power 2 both sides of this equation we get $4x-y=4$. Using by the same idea for the second equation of the given system we apply the formula (13) and obtain

$$\begin{aligned} \sqrt{y+\sqrt{x}}-\sqrt{y-\sqrt{x}} &= \frac{\left(\sqrt{y+\sqrt{x}}\right)^2-\left(\sqrt{y-\sqrt{x}}\right)^2}{\sqrt{y+\sqrt{x}}+\sqrt{y-\sqrt{x}}} \\ &= \frac{2\sqrt{x}}{\sqrt{y+\sqrt{x}}+\sqrt{y-\sqrt{x}}}=1 \end{aligned}$$

or

$$\sqrt{y+\sqrt{x}}+\sqrt{y-\sqrt{x}}=2\sqrt{x}. \tag{14}$$

After using (14) and the second equation of the given system we get the system

$$\left| \begin{array}{l} \sqrt{y+\sqrt{x}}-\sqrt{y-\sqrt{x}}=1 \\ \sqrt{y+\sqrt{x}}+\sqrt{y-\sqrt{x}}=2\sqrt{x} \end{array} \right. .$$

We add both equations of the last system and receive $2\sqrt{y+\sqrt{x}}=2\sqrt{x}+1$. After raising to a power 2 both sides of this equation we obtain $4x-4y=-1$. Then get the system of linear equations

$$\left| \begin{array}{l} 4x-y=4 \\ 4x-4y=-1 \end{array} \right. ,$$

whose solution is $x = \dfrac{17}{12}$ and $y = \dfrac{5}{3}$. It is easy to see that these roots are solutions of the given system.

II way. This problem can be solved in other way by using the formulas:

$$\sqrt{A+\sqrt{B}} = \sqrt{\frac{A+\sqrt{A^2-B}}{2}} + \sqrt{\frac{A-\sqrt{A^2-B}}{2}}, \tag{15}$$

$$\sqrt{A-\sqrt{B}} = \sqrt{\frac{A+\sqrt{A^2-B}}{2}} - \sqrt{\frac{A-\sqrt{A^2-B}}{2}}. \tag{16}$$

Applying formulas (15) and (16) we can write the system as:

$$\left| \begin{array}{l} \sqrt{\frac{x+\sqrt{x^2-y}}{2}} + \sqrt{\frac{x-\sqrt{x^2-y}}{2}} \\ \quad +\sqrt{\frac{x+\sqrt{x^2-y}}{2}} - \sqrt{\frac{x-\sqrt{x^2-y}}{2}} = 2 \\ \sqrt{\frac{y+\sqrt{y^2-x}}{2}} + \sqrt{\frac{y-\sqrt{y^2-x}}{2}} \\ \quad -\sqrt{\frac{y+\sqrt{y^2-x}}{2}} + \sqrt{\frac{y-\sqrt{y^2-x}}{2}} = 1 \end{array} \right.$$

We simplify and obtain

$$\left| \begin{array}{l} \sqrt{\frac{x+\sqrt{x^2-y}}{2}} = 1 \\ \sqrt{\frac{y-\sqrt{y^2-x}}{2}} = \frac{1}{2} \end{array} \right. .$$

We raise in the power 2 both sides of the equations in the last system and find

$$\left| \begin{array}{l} x+\sqrt{x^2-y} = 2 \\ y-\sqrt{y^2-x} = \frac{1}{2} \end{array} \right. \Leftrightarrow \left| \begin{array}{l} \sqrt{x^2-y} = 2-x \\ \sqrt{y^2-x} = y-\frac{1}{2} \end{array} \right. .$$

Again, we raise in the second degree, simplify and get the system

$$\left| \begin{array}{l} 4x-y=4 \\ 4y-4x=1 \end{array} \right. ,$$

whose solutions are $x = \dfrac{17}{12}$ and $y = \dfrac{5}{3}$.

Comparing these two ways, we see that the solution using the formula (12) is a shorter and easier than another solution, because there is no need to remember complex formulas such as (15) and (16).

In the following tasks, we will use the formula (8).

Task 5. Solve the equation $\sqrt{3x^2-5x+7}+\sqrt{3x^2-7x+2}=3$.

Solution: The domain of this equation is the set $D=\left(-\infty,\frac{1}{3}\right]\bigcup[2,\infty)$. Applying the formula (8) we get

$$\frac{3x^2-5x+7-3x^2+7x-2}{\sqrt{3x^2-5x+7}-\sqrt{3x^2-7x+2}}$$

or

$$\sqrt{3x^2-5x+7}-\sqrt{3x^2-7x+2}=\frac{2x+5}{3}. \tag{17}$$

Adding (17) to the given equation we get $\sqrt{3x^2-5x+7}=\dfrac{x+7}{3}$, but the expression of its left side is non-negative, then $x\geq-7$. We raise in a power 2 and find $26x^2-59x+14=0$, whose roots are the numbers 2 and $\dfrac{7}{26}$.

Task 6. Solve the equation

$$\sqrt{3x^2-7x+3}-\sqrt{3x^2-5x-1}=\sqrt{x^2-2}-\sqrt{x^2-3x+4}.$$

Solution: We apply the formula (7) for both sides of the given equation. It accepts the type

$$\frac{3x^2-7x+3-3x^2+5x+1}{\sqrt{3x^2-7x+3}+\sqrt{3x^2-5x-1}}=\frac{x^2-2-x^2+3x-4}{\sqrt{x^2-2}+\sqrt{x^2-3x+4}}.$$

After simplifying we get the equation

$$\frac{-2\,(x-2)}{\sqrt{3x^2-7x+3}+\sqrt{3x^2-5x-1}}=\frac{3\,(x-2)}{\sqrt{x^2-2}+\sqrt{x^2-3x+4}},$$

for which $x_1=2$. It can be proven that the given equation does not have other roots.

The following tasks are related by application of the formula (5).

Task 7. Calculate $\sqrt[3]{50+19\sqrt{7}}$.

Solution: This task can be solved in the following ways:

I way. We will represent $\sqrt[3]{50+19\sqrt{7}}$ in the form

$$\sqrt[3]{50+19\sqrt{7}}=\left(a+b\sqrt{7}\right), \tag{18}$$

where $a\neq0$, $b\neq0$ and we will raise both sides of the (18) in the power 3, after which we receive $50+19\sqrt{7}=a^3+3\sqrt{7}a^2b+21ab^2+7\sqrt{7}b^3$ or $50+$

$19\sqrt{7} = a^3 + 21ab^2 + \sqrt{7}\left(3a^2b + 7b^3\right)$.

From here, the task comes down to solving the system

$$\left| \begin{array}{l} a^3 + 21ab^2 = 50 \\ 3a^2b + 7b^3 = 19 \end{array} \right. ,$$

and then after division of both equations we get

$$\frac{a^3 + 21ab^2}{3a^2b + 7b^3} = \frac{50}{19}.$$

If the numerator and the denominator of the fraction in the left side are divided by b^3 and after substituting $\frac{a}{b} = z$, we will obtain

$$\frac{z^3 + 21z}{3z^2 + 7z^3} = \frac{50}{19}$$

or $19z^3 - 150z^2 + 399z - 350 = 0$. One of the roots of the cubic equation is $z = 2$. It can be proved that the other roots of this cubic equation are not real numbers. Because $\frac{a}{b} = z$, it follows that $a = 2b$. Then the first equation of the system has the form $8b^3 + 42b^3 = 50 \Rightarrow b = 1$ and $a = 2$. Then $\sqrt[3]{50 + 19\sqrt{7}} = 2 + \sqrt{7}$.

II way. We will use the formula (5). Let denote

$$\sqrt[3]{50 + 19\sqrt{7}} + \sqrt[3]{50 - 19\sqrt{7}} = x.$$

Then we raise the equation in the third degree and we get

$$50 + 19\sqrt{7} + 50 - 19\sqrt{7} + 3\sqrt[3]{-27}x = x^3$$

or $x^3 + 9x - 100 = 0$. This equation has only one real root $x = 4$. Therefore $\sqrt[3]{50 + 19\sqrt{7}} + \sqrt[3]{50 - 19\sqrt{7}} = 4$. Denote $\sqrt[3]{50 + 19\sqrt{7}} = y$, $y > 0$. Since $\sqrt[3]{50 + 19\sqrt{7}}.\sqrt[3]{50 - 19\sqrt{7}} = -3$, then $\sqrt[3]{50 - 19\sqrt{7}} = -\frac{3}{y}$. To calculate the value of the $\sqrt[3]{50 + 19\sqrt{7}}$ it is necessary to solve the equation $y - \frac{3}{y} = 4$ or $y^2 - 4y - 3 = 0$, whose roots are $y_{1/2} = 2 \pm \sqrt{7}$. However, since $y > 0$, then $\sqrt[3]{50 + 19\sqrt{7}} = 2 + \sqrt{7}$, but $\sqrt[3]{50 - 19\sqrt{7}} = 2 - \sqrt{7}$. The proposed solutions give us reason to argue that the second way is more rational. Therefore, students should pay attention of the question: how the application of these formulas leads to a shorter solution of the problems.

We will also illustrate the application of the formula (6).

Task 8. Solve the equation $\sqrt[3]{5x+7} - \sqrt[3]{5x-12} = 1$.

Solution: I way. By using the formula (6) we get

$$\begin{aligned} & 5x + 7 - 5x + 12 - 3\sqrt[3]{(5x+7)(5x-12)} = 1 \Leftrightarrow \\ & \Leftrightarrow \sqrt[3]{(5x+7)(5x-12)} = 6 \Leftrightarrow \\ & \Leftrightarrow 25x^2 - 25x - 300 = 0 \end{aligned}$$

or $x^2 - x - 12 = 0 \Rightarrow x_1 = -3$ and $x_2 = 4$.

II way: We substitute $\sqrt[3]{5x+7} = u$ and $\sqrt[3]{5x-12} = v$. Then we can write the system

$$\left| \begin{array}{l} u - v = 1 \\ u^3 - v^3 = 19 \end{array} \right. \quad \text{or} \quad \left| \begin{array}{l} u - v = 1 \\ u^2 + uv + v^2 = 19 \end{array} \right. .$$

If $v = u - 1 \Rightarrow u^2 + u(u-1) + (u-1)^2 = 19$ or $u^2 - u - 6 = 0$, whose roots are $u_1 = -2$ and $u_2 = 3$. However, since $\sqrt[3]{5x+7} = u$, then $x_1 = -3$ and $x_2 = 4$.

We will give one more formula

$$\begin{aligned} a^4 + b^4 &= \left(a^2 + b^2\right)^2 - 2a^2b^2 \\ &= \left[(a+b)^2 - 2ab\right]^2 - 2a^2b^2 \end{aligned} \tag{19}$$

and its application.

Task 9. Solve the equation $\sqrt[4]{13x+1} + \sqrt[4]{4x-1} = 3\sqrt[4]{x}$.

Solution: The domain of the given equation is $x \geq \frac{1}{4}$. We divide the both sides of the equation into $\sqrt[4]{x}$. Then the given equation gets the form $\sqrt[4]{13 + \frac{1}{x}} + \sqrt[4]{4 - \frac{1}{x}} = 3$. We substitute $\sqrt[4]{13 + \frac{1}{x}} = u$ and $\sqrt[4]{4 - \frac{1}{x}} = v$, and then we get the system

$$\left| \begin{array}{l} u + v = 3 \\ u^4 + v^4 = 17 \end{array} \right. ,$$

where $u \geq 0$ and $v \geq 0$. For the second equation, we apply (19) and we get $(9 - 2uv)^2 - 2u^2v^2 = 17 \Leftrightarrow u^2v^2 - 18uv + 32 = 0$, from where we find that $(uv)_1 = 2$ and $(uv)_2 = 16$. At first we have to decide the systems

$$\left| \begin{array}{l} u + v = 3 \\ uv = 2 \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} u + v = 3 \\ uv = 16 \end{array} \right. .$$

We find that the first one has two solutions the ordered pairs $(1; 2)$

and $(2; 1)$, and the second has no solution. From here we get two equations for the variable x, correspondingly $\sqrt[4]{13+\frac{1}{x}}=1$, $\sqrt[4]{13+\frac{1}{x}}=2$ with roots the numbers $-\frac{1}{12}$ and $\frac{1}{3}$. However, since $x \geq \frac{1}{4}$, then only $\frac{1}{3}$ is a solution to the given equation.

Task 10. Solve the equation $\sqrt[4]{10+x^2+x}+\sqrt[4]{7-x^2-x}=3$.

Solution: After substitution $\sqrt[4]{10+x^2+x}=u$ and $\sqrt[4]{7-x^2-x}=v$, we get the system

$$\left| \begin{array}{l} u+v=3 \\ u^4+v^4=17 \end{array} \right. ,$$

where $u \geq 0$ and $v \geq 0$. From here this task is solved in a similar way as a task 9. We obtained that the solution to the given equation are the numbers -3 and 2.

The consistent implementation of the actions in the process of searching for the solution of a given task forms in the student important skills with these features, which we talked about at the beginning of the article.

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