WORKING WITH MATHEMATICALLY GIFTED STUDENTS AGED 15–16

Valentina Gogovska, Katerina Anevska, Risto Malcheski

Abstract. *Working with gifted students is of special social interest. However, in the existing secondary education curricula, it is only declaratively mentioned and this important segment of the education system is completely neglected. That is why in this paper an attempt was made to develop a program for work with mathematically gifted students aged 15–16, i.e. for the first year of secondary education.*

Key words: gifted students, curricula, aims and objectives of the curriculum.

1. Introduction

Papers [1, 2, 3] and [6] provide curricula for working with mathematically gifted students from 6th to 9th grade in the nine-year primary education, i.e. for students aged 11–15 years. Having in mind the social significance of working with mathematically gifted students and the fact that as a rule it is realized turbulently, in this paper we will provide an integral curriculum for working with mathematically gifted students aged 15 – 16 years. We believe that the preparation of such a curriculum, as well as the preparation of appropriate books and collections of problems that are complementary to the curriculum, will greatly fill the gap caused by society's neglect of the development of these children. Namely, in almost all countries, without exception, there is a formal support for the progress of these children, most often expressed through the organization of competitions and awarding of scholarships, and practically doing nothing further for the development of these children. At the same time, in some countries there are so-called mathematics gymnasiums, which try to fill the gap in the work with mathematically gifted students. However, we seem to forget that these schools must not be places that prepare students for participation in prestigious math competitions (JBMO, BMO, APMO, EGMO, IMO, etc.), but places that direct students through specialized programs at a higher level towards natural and technical sciences.

This paper is in a way a continuation of the above mentioned papers. In addition, based on the experience of the authors, but also the experience of countries in the immediate and wider surrounding, an attempt was made for part of the topic Number Theory to give an example of a system of problems that would determine the level that students should reach at this age. Finally, the listed references give several collections of problems that are intended for working with gifted students in secondary education and which are part of a system of thirty such collections made by the authors and their collaborators.

2. Program for working with mathematically gifted students aged 15–6

In this section, we will present an integral curriculum for working with mathematically gifted students aged 15–16, that is to say for students in the first year of secondary education. The offered curriculum actually builds on the respective teaching curricula that were previously made for students in grades 6–9 in primary education and are presented in papers [1, 2, 3, 6]. During the preparation of the curricula, the method of concentric circles was used, which means that part of the contents that were adopted in the previous years at a certain level are expanded and extended. This curriculum should be implemented continuously, and not only in periods when students are preparing for certain math competitions. The goals of the curriculum for students aged 15–16 are:

- To develop students' qualities of thinking such as: flexibility, stereotyping, width, rationality, depth and criticality,
- To strive for the student to adopt scientific methods: observation, comparison, experiment, analysis, synthesis, classification, systematization and axiomatic method,
- To strive for the student to adopt the types of conclusions: induction, deduction and analogy, whereby it is of particular importance to present suitable examples from which the student will realize that the analogy conclusion is not always correct,
- The student to adopt the prescribed contents in the field of polynomials and to enable them to apply the same when solving appropriate problems,
- The student to adopt the prescribed contents in the field of com-

binatorics and to enable them to use the same,

- The student to adopt the prescribed contents in the field of Number theory and to enable them to apply the same when solving appropriate problems,
- The student to adopt the prescribed contents in the field of inequalities and to enable them to use the same,
- The student to adopt the prescribed contents in the field of polynomials and to enable them to use the same,
- The student to adopt the prescribed contents in the field of geometry and to enable them to use the same,
- The student to adopt the prescribed contents in the field of mathematical induction and to enable them to apply the same.

In order to achieve the aforementioned goals, it is necessary to adopt the following contents:

***Algebra* (4 classes per week – 144 classes in a school year).** *Logic*: statements, conjunction, disjunction, negation, implication and equivalence, propositional formulas, tautology, methods for proving tautologies, completeness of statement logic, predicates and quantifiers, rules for deduction, methods for proving mathematical propositions, solving logical problems; *Sets and combinatorics*: sets, set operations and relations, proving basic set identities, mapping, injection, surjection and bijection, composition of mappings (properties) invertible mapping (function) and its properties, equivalent sets, principle of equality, principle of sum, product, inclusion and exclusion, Dirichlet's principle, *Binary relations*: binary relation, reflexive, symmetric, transitive and antisymmetric, equivalence relation and order relation; *Natural numbers and integers*: natural numbers, Peano's axioms, operations with natural numbers, principle of mathematical induction, forward-backward induction, integers, operations with integers; *Number theory*: divisibility in the set of integers, division with remainder, general and special signs of divisibility, greatest common divisor, Euclidean algorithm and least common denominator, notion of prime and complex number, the sieve of Eratosthenes, infinity of primes, fundamental theory of arithmetic, Fermat and Mersenne's primes, functions $y = [x]$, $y = \{x\}$ and their application in Number theory, linear Diophantine's equation, Euler's method, elementary methods for solving nonlinear Diophantine's equations, notion of congruence, basic properties of congruences, application of congruences, classes of congruences, linear congruent equation; *Real numbers*: groupoid, subgroup, subgroupoid, subsemigroup, neutral and inverse element, notion of group, subgroup (examples), ring and field, ordered field, field of rational numbers, irrational numbers, field of real numbers; *Inequalities*: properties of the inequality relations $\leq$, $<$ in the set of real numbers, proving basic inequalities, proving inequalities by induction, arithmetic, geometric, harmonic and quadratic mean, inequalities between them and application, *Polynomials and algebraic rational expressions*: polynomials, definition, equality of polynomials, addition, subtraction, multiplication and division of polynomials, Bzout's theorem, formulas for difference of squares, sum and difference of cubes, square and cube of binomial, Euclidean algorithm for polynomials, polynomials with real numbers as coefficients, factoring polynomials, LCM and GCD of polynomials, notion of algebraic rational expression, addition, subtraction, multiplication, and division of algebraic rational expressions, transformation of algebraic rational expressions; *Linear function, equation and inequality*: linear function and its graph, graph of the function $y = |x|$, problems with linear function, the equation $ax + b = 0$, discussion and problems with linear equations, linear equation with absolute values and parametric linear equations, problems with one variable, system of two linear equations with two variables, discussion, problems with two variables, system of three linear equations with two variables, discussion, problems with three variables, system of linear inequalities, graphic interpretation; *Power and power function*: raising to the power, definition and properties, the function $y = x^n$, $n \in \mathbb{N}$ raising to the integer exponent, function $y = x^{-n}$, $n \in \mathbb{N}$, transformation of expressions with power function, notion of square root of a real number, properties of square roots, the notion of n-th root of a real number, raising to a rational exponent and properties of raising to a rational exponent, transformation of irrational expressions.

Geometry **(4 classes per week 144 classes in a school year).** *Introduction to geometry*: Deductive and inductive methods of inference, axiomatic foundation of geometry, basic concepts and basic claims in geometry, incidence axiom and consequences, ordering axioms and consequences, definition of a line-segment, rays, half-plane, angle, congruence axioms, axiom of continuity, parallel axiom (Playfair's axiom) and consequences; *Congruence*: isometric transformations, congruence of plane figures, congruence of line-segments, congruence of angles, right angles, the relation of perpendicular lines, congruence of triangles, angles of a transversal, sum of angles of a triangle, inequality of triangle, quadrilateral, parallelogram, midline of a triangle, centers of a triangle (circumcenter, incenter, centroid and orthocenter), Euler's circle (nine-point circle), Steiner's theorem, Sylvester's problem; *Vectors*: notion of vectors, vector addition, component of a vector, multiplying a vector with a rational number, partitioning a line-segment in a given ratio, multiplication of vector with real number, Thales' theorem and converse of Thales' theorem, vector applications in Euclidean geometry (midline of a triangle, trapeze and the like), Hamilton's theorem, Euler's line, linear dependence of vectors, basis for a vector space, decomposition of a vector, operations with vectors in a coordinate system; *Application of congruence*: application of disk and circle congruence, central and inscribed angle of a circle, properties, tangent to a circle, tangential quadrilateral, chordal quadrilateral, Miquel point for a triangle, Simson's line and Simson's theorem, Torricelli's point for a triangle, the relation line perpendicular to a plane, dihedron, perpendicular planes, angle between a line and a plane, angle between two skew lines; *Construction*: solving construction problems, construction of plane figures with the help of previously adopted properties of a triangle, square and circle, isometric plane transformations and their classification, direct and indirect isometric transformations, involution, axial symmetry, representing plane isometry with axial symmetries, pencil of lines in a plane, central rotation, central symmetry, translation, glide-reflection symmetry, classification of plane isometries, Chasles' theorem; *Similarity*: ratio of segments, Thales's theorem, homothety, properties, composition of homotheties, similarity transformation, similar shapes, similarity of triangles, Euler's formula, length of a median, length of an internal bisector of an angle of a triangle, projective harmonic conjugate, Apollonius' circle, Ceva's theorem, Menelaus' theorem, theorems of Ptolemy, Stewart, Leibniz, Desargues, Pascal, Pappus, Morley and Carnot, Ptolemy's inequality, the Pythagorean theorem, Gergonne point, Nagel point, Lemoine point for a triangle (symmedian point), the butterfly theorem, Euler's quadrilateral theorem, radical axis and pencil of circles, power center, golden ratio; *Inversion*: notion and basic properties of inversion, Apollonius' circles, Mikel's theorem for six circles, Feuerbach's points for a triangle; *Trigonometry of a right-angled triangle*: trigonometric functions of an acute angle, values of trigonometric functions of some acute angles, important trigonometric identities, trigonometric functions of a complementary angle, solving a right-angled triangle.

3. Example of a system of problems from the section on Number theory

In order to realize the suggested curricula for working with gifted 15–16 years old students, it is necessary to make appropriate teaching aids, that is to say, textbooks that must be accompanied by appropriate books with collections of problems. Therefore, we will present a system of problems for the functions $y = [x]$, $y = \{x\}$ and their application in the Number theory, which we deem will help student adopt knowledge and skills to achieve the curriculum's goals at a highest possible level.

1. Prove that for each natural number $k \geq 2$ there is a real number $x \neq 0$ such that $k = \dfrac{[x]\{x\}}{x}$.
2. Prove that $x + \left[\dfrac{n}{x}\right] \geq 2[\sqrt{n}]$, for every x, $n \in \mathbb{N}$.
3. Prove that if $\dfrac{p}{q}$, $p, q \in \mathbb{N}$ is not an integer, then $\dfrac{p}{q} \geq \left[\dfrac{p}{q}\right] + \dfrac{1}{q}$.
4. Determine the number of natural numbers that are not greater than 2021 and are $[\sqrt{n}]\,|n$.
5. Which terms in the sequence $a_n = \left[\sqrt{n(n+2)(n+4)(n+6)}\,\right]$, $n \in \mathbb{N}$ are divisible by 7?
6. Prove that for every positive number x, $[\sqrt{[\sqrt{x}]}] = [\sqrt{\sqrt{x}}]$ is valid.
7. If natural numbers x and y satisfy the equality
$$[(4 + 2\sqrt{3})x] = [(4 - 2\sqrt{3})y],$$
prove that they are of opposite parity.
8. Prove that for every real number a and for every natural number n , $n[a] \leq [na] \leq n[a] + n - 1$ is valid.
9. Prove that $[a[na]]+1 = [na^2]$ is valid for every $n \in \mathbb{N}$ and $a = \frac{1+\sqrt{5}}{2}$.
10. Determine the positive real numbers a so that for every $n \in \mathbb{N}$, $[a[(a + 1)n]] = n - 1$ is valid.
11. Let $a = \dfrac{3 + \sqrt{5}}{2}$. Prove that $3|[a[an]] + n$, for every $n \in \mathbb{N}$.
12. If some real number x satisfies the equality $\{8x\} = \{15x\}$, then $\{26x\} = \{75x\}$ is valid. Prove it!

13. Is there a natural number n such that the fractional part of the number $(2+\sqrt{2})^n$ is grater than 0,999999?
14. Prove that for every natural number n, the number $[(2-\sqrt{3})^n]$ is an odd natural number.
15. Prove that the terms of the sequence $\{10^n\sqrt{2}\},\ n=0,1,2,...$ are of opposite parity.
16. Let $a_n = n\sqrt{5}-[n\sqrt{5}],\ n\in\mathbb{N}$. Determine the lowest and highest number between the numbers $a_1, a_2, ..., a_{2009}$.
17. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined with: $a_1=2,\ a_{n+1}=\left[\frac{3}{2}a_n\right]$, for $n=1,2,....$ Prove that in the sequence $\{a_n\}_{n=1}^{\infty}$ there are infinitely many odd and infinitely many even numbers.
18. a) Give an example for a number a such that $\{a\}+\{\frac{1}{a}\}=1$.
    b) Prove that such number a cannot be rational.
19. In the set $\mathbb{R}$ solve the equation: $[x[x]]=1$.
20. Solve the equation: $\{x\}+\{2x\}+\{3x\}=x$.
21. Solve the equation: $x^2-2[x]+\{x\}=0$.
22. A natural number is given n. Determine the number of solutions of the equation $x^2-[x^2]=(x-[x])^2$ such that $1\le x\le n$.
23. Solve the equation: $x^3-[x]=4$.
24. Determine all the real numbers a such that $4[an]=n+[a[an]]$, for every natural number n.
25. Determine all the natural numbers n for which $n-[n\{n\}]=2$.
26. Determine all the pairs (a,b) real numbers for which $a[bn]=b[an]$ is valid, for every natural number n.
27. Determine the number of different terms of the finite sequence with general term $\left[\frac{k^2}{1998}\right]$, where $k=1,2,...,1997$.
28. Determine all the real numbers $x>1$ such that $\sqrt[n]{[x^n]}$ is a natural number for every $n\in\mathbb{N}\backslash\{1\}$.
29. Determine the integer part of the number

$$a_n=\sqrt[3]{24+\sqrt[3]{24+...+\sqrt[3]{24}}},\quad (n \text{ roots}),$$

where $n \geq 1$

30. How many of the first 1000 positive numbers can be written down in the form $[2x] + [4x] + [6x] + [8x]$, where x is a real number?
31. Let $n \in \mathbb{N}$. Prove that 2^n is not divisor of $n!$.
32. What is the exponent of the prime number p in the canonic factorization of $p^n!$?
33. Prove that $\dfrac{(ab)!}{a!(b!)^a}$ is a natural number, for each $a,\ b\ \in \mathbb{N}$.
34. Calculate the sums:

    a) $\displaystyle\sum_{k=1}^{n^2+2n} k[\sqrt{k}]$,

    b) $\displaystyle\sum_{k=1}^{n^2} \frac{n - [\sqrt{k-1}]}{\sqrt{k} + \sqrt{k-1}}$.
35. Prove that $\displaystyle\sum_{k=1}^{n^2-1} [\sqrt{k}] = \frac{1}{6}(n-1)n(4n+1)$, for every $n \in \mathbb{N}$.
36. For every real number x prove that
$$[x] + \frac{[2x]}{2} + \frac{[3x]}{3} + ... + \frac{[nx]}{n} \leq [nx].$$
37. Prove that for every rational number α there can always be an interval $[c, d] \subset [0, 1]$, that does not contain any of the terms of the sequence $a_n = \{\alpha n\},\ n = 1, 2, 3, ...$
38. Prove that for every irrational number α in every interval $(c, d) \subset (0, 1)$, there is at least one term of the sequence $a_n = \{\alpha n\},\ n = 1, 2, ...$
39. Prove that the number sequence $[n\sqrt{2}]$, $n \in \mathbb{N}$ contains infinitely many powers of the number 2.
40. Prove that for every real number $\alpha > 0$ there are infinitely many natural numbers n, such that $n^2 + 1 | [\alpha n]$.

4. Conclusion

Continuous work with mathematically gifted students is a basic prerequisite for their faster progress. It should take place according to a curriculum appropriate for the age of the students, which we have previously presented for students aged 15–16. Among other things, we believe that the acquisition of theoretical knowledge provided by this curriculum, supported by the collections of problems listed in the presented references will allow:

- improving the qualities of students' thinking, and especially the depth, width and critical thinking in general,
- students at a higher level to adopt the inductive, deductive and axiomatic scientific method, as well as inductive and deductive reasoning,
- to enable students to make significant use of elementary laws of logic in problem solving, as well as deduction rules modus ponens, modus tollens, contraposition rules, double negation, hypothetical syllogism, and De Morgan's laws.

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Valentina Gogovska[1,*], Katerina Anevska[2], Risto Malcheski[3]
[1,*] Faculty of Natural Sciences and Mathematics,
Ss. Cyril and Methodius University, Skopje, Macedonia
[2] FON University, Skopje, Macedonia
anevskak@gmail.com
[3] International Slavic University "G. R. Derzhavin" Sv. Nikole, Macedonia
risto.malceski@gmail.com
[*] Corresponding author: valentinagogovska@gmail.com