ABOUT THE LOCATION OF THE ROOTS OF ONE KIND THIRD DEGREE POLYNOMIALS WITH COEFFICIENTS DEPENDING ON THE VERTICES OF QUADRILATERALS

Sava Grozdev, Veselin Nenkov

Abstract. *A class of third degree polynomials is considered with roots depending on the vertices of a quadrilateral. A geometric relation is derived concerning the roots of such polynomials and the foci of the quadrilateral in-conics and the intersection point of its diagonals. At the end, polynomials of a real variable and with real coefficients are considered as an application.*

Key words: polynomial, roots of a polynomial, quadrilateral, ellipse, focus.

A geometric relation is described in [6] concerning one type of polynomials with roots in the vertices of a convex quadrilateral and the roots of the corresponding derivatives. The relation is contained in the following assertion:

Theorem 1.1. *Let* k_j $(j = 1, 2, 3, 4)$ *be positive integers and* k *be an ellipse which is inscribed in the convex quadrilateral* $A_1A_2A_3A_4$ *in such a way that for the segments* A_1A_2, A_2A_3, A_3A_4, A_4A_1 *and the corresponding tangent points* P_1, P_2, P_3, P_4 *on* k *the following equations are verified:*

$$\overline{A_2P_2} : \overline{A_3P_2} = -k_2 : k_3,$$
$$\overline{A_3P_3} : \overline{A_4P_3} = -k_3 : k_4,$$
$$\overline{A_4P_4} : \overline{A_1P_4} = -k_4 : k_1.$$

If a polynomial $P(z)$ *of a complex variable, of degree* $n = k_1 + k_2 + k_3 + k_4$ *and with complex coefficients has a* k_j*- multiple root in the vertex* A_j $(j = 1,\ 2,\ 3,\ 4)$ *of* $A_1A_2A_3A_4$, *then the derivative of* $P(z)$ *has roots in the point* $P_0 = A_1A_3 \cap A_2A_4$ *and in the foci of the ellipse* k.

It follows from this theorem that the mentioned geometric relation contains the intersection point of the diagonals of the quadrilateral under consideration and the foci of a suitable ellipse, which is inscribed in the quadrilateral. In addition, the tangent points divide the sides of the quadrilateral in rational simple quotients. From one side, ellipses could be inscribed in the quadrilateral with tangent points dividing the sides in irrational simple quotients. From another side, the quadrilateral has ex-conics, some of which are ellipses too. A question could be raised for the existence of polynomials with coefficients depending on the vertices of a quadrilateral (not convex obligatorily) and an arbitrary conic, which is inscribed in it. From its part the location of the roots of such polynomials should be the same as the one of the roots of the derivative in the mentioned theorem.

Such polynomials could be realized by means of functions of the form

$$f\left(z\right) = a_0(z-a_1)^{k_1}(z-a_2)^{k_2}(z-a_3)^{k_3}(z-a_4)^{k_4},$$

where a_0, a_1, a_2, a_3, a_4 are complex numbers, while k_1, k_2, k_3 and k_4 are real numbers different from zero. The derivative of this function could be presented in the form:

$$f'\left(z\right) = a_0(z-a_1)^{k_1-1}(z-a_2)^{k_2-1}(z-a_3)^{k_3-1}(z-a_4)^{k_4-1}.P\left(z\right),$$

where

$$\begin{aligned} P\left(z\right) = {} & k_1\left(z-a_2\right)\left(z-a_3\right)\left(z-a_4\right) + k_2\left(z-a_3\right)\left(z-a_4\right)\left(z-a_1\right) + \\ & + k_3\left(z-a_4\right)\left(z-a_1\right)\left(z-a_2\right) + k_4\left(z-a_1\right)\left(z-a_2\right)\left(z-a_3\right). \end{aligned}$$

The function $P\left(z\right)$ is a polynomial of degree 3 at most and could be presented in the form

$$P\left(z\right) = az^3 - bz^2 + cz - d, \tag{1}$$

where

$$\begin{aligned} a = {} & k_1 + k_2 + k_3 + k_4, \\ b = {} & \left(a_2 + a_3 + a_4\right)k_1 + \left(a_3 + a_4 + a_1\right)k_2 + \\ & + \left(a_4 + a_1 + a_2\right)k_3 + \left(a_1 + a_2 + a_3\right)k_4, \\ c = {} & \left(a_2a_3 + a_3a_4 + a_4a_2\right)k_1 + \left(a_3a_4 + a_4a_1 + a_1a_3\right)k_2 + \\ & + \left(a_4a_1 + a_1a_2 + a_2a_4\right)k_3 + \left(a_1a_2 + a_2a_3 + a_3a_1\right)k_4, \\ d = {} & a_2a_3a_4k_1 + a_3a_4a_1k_2 + a_4a_1a_2k_3 + a_1a_2a_3k_4. \end{aligned}$$

Polynomials of the form (1) solve the raised question, when a_1, a_2, a_3, a_4 are the affixes of the vertices of the quadrilateral $A_1A_2A_3A_4$. The solution is expressed by the next:

Theorem 1.2. *Let the lines A_1A_2, A_2A_3, A_3A_4 and A_4A_1 be tangent to the conic k at the points P_1, P_2, P_3 and P_4, respectively defining the quadrilateral $A_1A_2A_3A_4$, while the real numbers k_1, k_2, k_3 and k_4 be such that the* *following equations are verified:*

$$\begin{aligned} \overline{A_1P_1} : \overline{A_2P_1} = -k_1 : k_2, \quad \overline{A_2P_2} : \overline{A_3P_2} = -k_2 : k_3, \\ \overline{A_3P_3} : \overline{A_4P_3} = -k_3 : k_4, \quad \overline{A_4P_4} : \overline{A_1P_4} = -k_4 : k_1. \end{aligned} \tag{2}$$

1) *If k is an ellipse or a hyperbola, then the roots of the polynomial (1) are located in the foci of k and the point $U = A_1A_3 \cap A_2A_4$;*

2) *If k is a parabola, then $k_1 + k_2 + k_3 + k_4 = 0$, while the roots of the polynomial (1) are located in the focus of k and the point $U = A_1A_3 \cap A_2A_4$.*

Proof. The proof of this theorem could be elaborated in the following way: Let the focus of the ellipse k be in the point O, let the focal parameter be p and the number eccentricity be e. As shown in [5] and [7], with respect to the Gaussian coordinate system K_0 from Figure 1 the affixes p_j and a_j of the points P_j and A_j $(j = 1, 2, 3, 4)$ are expressed by the formulae, respectively

$$\begin{aligned} p_1 = \frac{2p}{e.t_1^2 + 2t_1 + e}, \quad p_2 = \frac{2p}{e.t_2^2 + 2t_2 + e}, \\ p_3 = \frac{2p}{e.t_3^2 + 2t_3 + e}, \quad p_4 = \frac{2p}{e.t_4^2 + 2t_4 + e}. \end{aligned} \tag{3}$$

$$\begin{aligned} a_1 = \frac{2p}{et_4t_1 + t_4 + t_1 + e}, \quad a_2 = \frac{2p}{et_1t_2 + t_1 + t_2 + e}, \\ a_3 = \frac{2p}{et_2t_3 + t_2 + t_3 + e}, \quad a_4 = \frac{2p}{et_3t_4 + t_3 + t_4 + e}, \end{aligned} \tag{4}$$

where $|t_1| = |t_2| = |t_3| = |t_4| = 1$.

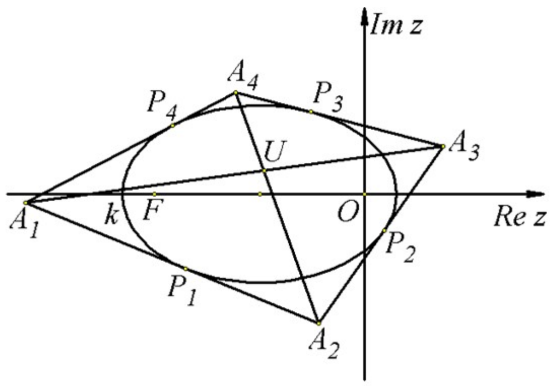

Figure 1.

Now, we will find dependencies between the parameters under consideration aiming at fulfillment of the equations (2). Since the simple quotient of arbitrary three points A_k, A_l and P_j from one line is expressed by the equation $\dfrac{\overline{A_k P_j}}{\overline{A_l P_j}} = \dfrac{a_k - p_j}{a_l - p_j}$, the following equations could be derived from (3) and (4) after some not complicated computations

$$\begin{aligned} \frac{\overline{A_1P_1}}{\overline{A_2P_1}} &= \frac{t_4 - t_1}{t_2 - t_1}.\frac{a_1}{a_2}, & \frac{\overline{A_2P_2}}{\overline{A_3P_2}} &= \frac{t_1 - t_2}{t_3 - t_2}.\frac{a_2}{a_3}, \\ \frac{\overline{A_3P_3}}{\overline{A_4P_3}} &= \frac{t_2 - t_3}{t_4 - t_3}.\frac{a_3}{a_4}, & \frac{\overline{A_4P_4}}{\overline{A_1P_4}} &= \frac{t_3 - t_4}{t_1 - t_4}.\frac{a_4}{a_1}. \end{aligned} \tag{5}$$

From (5) we obtain

$$a_2 = -\frac{k_2}{k_1}.\frac{t_4 - t_1}{t_2 - t_1}.a_1, \quad a_3 = -\frac{k_3}{k_1}.\frac{t_4 - t_1}{t_3 - t_2}.a_1, \quad a_4 = -\frac{k_4}{k_1}.\frac{t_4 - t_1}{t_4 - t_3}.a_1. \tag{6}$$

It follows from (6) that

$$d = a_2a_3a_4k_1 + a_3a_4a_1k_2 + a_4a_1a_2k_3 + a_1a_2a_3k_4 = 0.$$

Consequently, $z = 0$ is a root of $P(z)$. Thus, it is proved that one of the foci O of k is a root of $P(z)$. If k has a second focus F, the proof that it is a root of $P(z)$ could be obtained moving the coordinate origin to F.

Now, we will show that the point U is a root of $P(z)$. Since $\overline{A_3U} : \overline{A_1U} = -k_3 : k_1$ and $\overline{A_4U} : \overline{A_2U} = -k_4 : k_2$, then $\overrightarrow{OU} = \dfrac{k_3\overrightarrow{OA_1} + k_1\overrightarrow{OA_3}}{k_3 + k_1}$ and $\overrightarrow{OU} = \dfrac{k_4\overrightarrow{OA_2} + k_2\overrightarrow{OA_4}}{k_4 + k_2}$. Using (6) for the affix of U, we obtain the next two equations:

$$\begin{aligned} u &= \frac{k_3 (t_1 - t_2 + t_3 - t_4)}{(t_3 - t_2)(k_3 + k_1)} a_1, \\ u &= \frac{k_2k_4 (t_4 - t_1)(t_1 - t_2 + t_3 - t_4)}{k_1 (t_1 - t_2)(t_3 - t_4)(k_2 + k_4)} a_1. \end{aligned} \tag{7}$$

Equalizing the values of u from (7) we obtain the dependence

$$\begin{aligned} &(k_1k_2k_3 + k_2k_3k_4 + k_3k_4k_1 + k_4k_1k_2)(t_3t_1 + t_4t_2) = \\ &= k_3k_1 (k_4 + k_2)(t_2t_3 + t_4t_1) + k_4k_2 (k_3 + k_1)(t_1t_2 + t_3t_4). \end{aligned} \tag{8}$$

From (1) and (7) we have

$$P(u) = \frac{k_3[k_1(t_2-t_3)-k_3(t_4-t_1)]^2(t_1-t_2+t_3-t_4)\,a_1^3}{k_1^2(k_3+k_1)^3(t_1-t_2)(t_2-t_3)^3(t_3-t_4)}\times$$
$$\times\left[(k_1k_2k_3+k_2k_3k_4+k_3k_4k_1+k_4k_1k_2)(t_3t_1+t_4t_2)-\right.$$
$$\left.-\,k_3k_1(k_4+k_2)(t_2t_3+t_4t_1)+k_4k_2(k_3+k_1)(t_1t_2+t_3t_4)\right].$$

It follows from here and the equation (8) that $P(u)=0$, which means that u is a root of $P(z)$. Thus, the assertion of the theorem is proved for the intersection point of the diagonals of $A_1A_2A_3A_4$ too.

It remains to prove that when k is a parabola, then $k_1+k_2+k_3+k_4=0$. From the equations (4) and (6) we obtain

$$k_2 = \frac{(t_2-t_1)(et_4t_1+t_4+t_1+e)\,k_1}{(t_1-t_4)(et_1t_2+t_1+t_2+e)},$$
$$k_3 = \frac{(t_3-t_2)(et_4t_1+t_4+t_1+e)\,k_1}{(t_1-t_4)(et_2t_3+t_2+t_3+e)},$$
$$k_4 = \frac{(t_4-t_3)(et_4t_1+t_4+t_1+e)\,k_1}{(t_1-t_4)(et_3t_4+t_3+t_4+e)}.$$

It follows from here that

$$k_1+k_2+k_3+k_4 = k_1\left(e^2-1\right)(t_1-t_3)(t_2-t_4)\times$$
$$\times\frac{[(t_1-t_2+t_3-t_4+t_1t_2t_3-t_2t_3t_4+t_3t_4t_1-t_4t_1t_2)+2(t_1t_3-t_2t_4)]}{(t_1-t_4)(et_1t_2+t_1+t_2+e)(et_2t_3+t_2+t_3+e)(et_3t_4+t_3+t_4+e)}.$$

When $e=1$ it follows from the last equation that $k_1+k_2+k_3+k_4=0$.

Thus, the theorem is proved.

□

If polynomials of a real variable and with real coefficients are considered, we could present some geometric interpretations of the proved theorem. Two ellipses k_P and k_Q are shown in Figure 2 inscribed in a deltoid. The corresponding polynomials are $P(x)$ and $Q(x)$. Figure 3 presents a hyperbola k_R and a parabola k_S, inscribed in the same deltoid. The corresponding polynomials are $R(x)$ and $S(x)$. The polynomial $S(x)$ is of second degree (its graphical representation is a parabola), as it is envisaged in the proved theorem. A hyperbola k_M and an ellipse k_L are shown in Figure 4 inscribed in a rhombus. The corresponding polynomials are $M(x)$ and $L(x)$. No parabola exists in this case. All cases are realized when $k_1 = 0,05$.

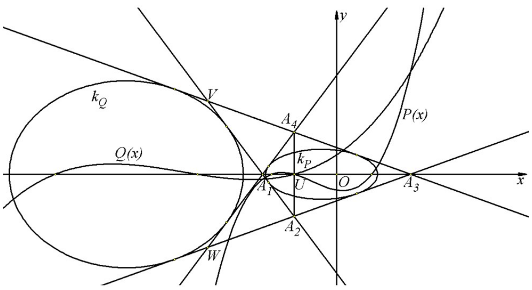

Figure 2.

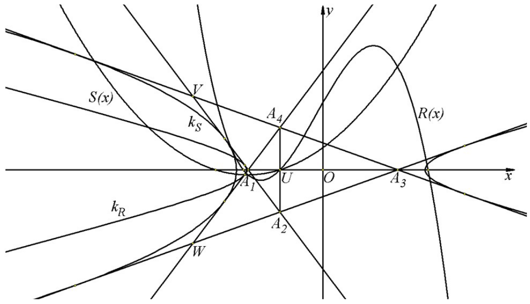

Figure 3.

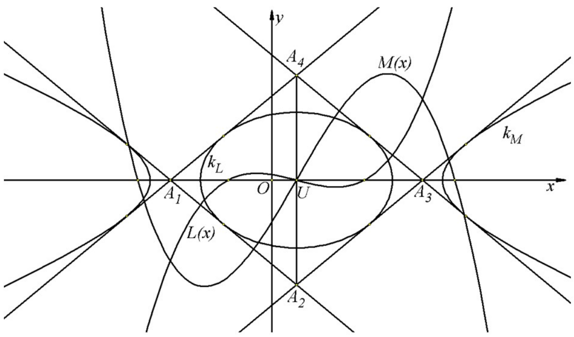

Figure 4.

References

[1] G. Genov, S. Mihovski, T. Mollov, *Algebra with number theory*, Sofia, Nauka i izkustvo, (1991).

[2] S. Grozdev, V. Nenkov, Polynomials of fourth degree with roots in the vertices of a parallelogram, *Mathematics and informatics*, (2018), **3**, 283–293, ISSN: 1310-2230.

[3] S. Grozdev, V. Nenkov, Polynomials with multiple roots in the vertices of a triangle, *Mathematics and informatics*, (2018), **4**, 352–359, ISSN: 1310-2230.

[4] S. Grozdev, V. Nenkov, Polynomials with multiple roots in the vertices of a parallelogram, *Mathematics and informatics*, (2019), **4**, 435–443, ISSN: 1310-2230.

[5] S. Grozdev, V. Nenkov, Several properties of the inscribed conic sections and a method for proofs with complex numbers, *International*

Journal of Computer Mathematics (IJCM), (2020), Vol. **5**, 53–70, ISSN: 2367-7775.

[6] S. Grozdev, V. Nenkov, On the polynomials with roots in the vertices of a class of convex quadrilaterals, *Mathematics and informatics*, (2020), **3**, 324–399, ISSN: 1310-2230.

[7] V. Nenkov, Conic sections, inscribed in a triangle, Mathematics and informatics, (1998), **5**, 54–59, ISSN: 1310-2230.

Sava Grozdev[1,*], Veselin Nenkov[2]
[1] Association for the Development of Education,
10, Saint Ekaterina Str., 1618 Sofia, Bulgaria,
[2] Department of Mathematics and Physics,
Faculty of Engineering, Nikola Vaptsarov Naval Academy,
73, Vasil Drumev Str., Varna, Bulgaria,
e-mail: vnenkov@mail.bg
[*] Corresponding author: sava.grozdev@gmail.com